Global Hypersurfaces of Section

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GLOBAL HYPERSURFACES OF SECTION FOR GEODESIC FLOWS ON CONVEX HYPERSURFACES

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ABSTRACT. We construct a global hypersurface of section for the geodesic flow of a convex hypersurface in Euclidean space admits an isometric involution. This generalizes the Birkhoff annulus to higher dimensions.

1. INTRODUCTION

Global surfaces of section, an idea introduced by Poincaré in his work on celestial mechanics [Poi87] and also explored by Birkhoff [Bir66], feature prominently in the literature on the 3-dimensional dynamics. They allow us to reduce the dynamics of vector fields on 3-manifolds to the dynamics of surface diffeomorphisms. Ghys [Ghy09] called the existence of a global surface of section as a paradise for dynamicists, since it eliminates technical difficulties and allows to investigate the pure nature of dynamical systems.

Main Result

Existence of global hypersurfaces of section of

1. symmetric mechanical Hamiltonian defined on $T^*\mathbb{R}^n$.
2. geodesic flow on symmetric convex hypersurface in $\mathbb{R}^{n+1}$.

Contents

1. **Basic notions**
   Hamiltonian and Reeb dynamics, global hypersurfaces of section and open book decompositions.
2. **Case 1. Mechanical Hamiltonians**
   Including harmonic oscillators, ellipsoids and Hènon-Heiles system
3. **Case 2. Geodesic Flow on Convex Hypersurfaces**
   Including hypersurfaces of revolution and Kepler problem

Basic Notions

Hamiltonian Dynamics

(W, ω): symplectic manifold, i.e. ω is nondegnerate closed 2-form.

$H : W \to \mathbb{R}$: **Hamiltonian** (which is just a smooth map)

Hamiltonian vector field is given by

$$i_{X_H}\omega = -dH.$$

The dynamics induced by X_H is called **Hamiltonian dynamics**.

Note. X_H is tangent to the regular level set $H^{-1}(c)$.

In other words, H is constant along the orbit of X_H

(Conservation of the mechanical energy)

Example : Mechanical Hamiltonian

Mechanical Hamiltonian ($=$ total mechanical energy) is given by

$$\begin{aligned} H : (T^*\mathbb{R}^n, dp \wedge dq) &\to \mathbb{R} \\ (q, p) &\mapsto \frac{1}{2}|p|^2 + V(q) \end{aligned}$$

The Hamiltonian vector field is given by

$$X_H = p \cdot \partial_q - \nabla V \cdot \partial_p.$$

The Hamiltonian flow equation is given by

$$\ddot{q} = -\nabla V.$$

This is Newton's second law of motion.

Reeb Dynamics

$(Y^{2n+1}, \ker\alpha)$: Contact manifold (α is 1-form s.t. $\alpha \wedge (d\alpha)^n \neq 0$)

Reeb vector field R is the unique vector field s.t. $\alpha(R) = 1$, $i_R d\alpha = 0$.

The dynamics induced by R is called **Reeb dynamics**.

Relationship between Hamiltonian and Reeb dynamics

$H : W \to \mathbb{R}$: Hamiltonian with regular value c.

Liouville vector field X : $\mathcal{L}_X \omega = \omega$ and positively transverse to $H^{-1}(c)$.

If there exists a Liouville vector field,

1. $(Y = H^{-1}(c), \ker(i_X \omega))$ is a contact manifold.
2. The Reeb vector field is parallel to X_H.
3. Flow of X_H and R are same up to reparametrization.
4. Bijection between sets of closed orbits of X_H and R.

Global Hypersurface of Section

Y: closed manifold, X: non-vanishing vector field on Y

A **global hypersurface of section** (simply GHS) of X is an embedded submanifold $P \subset Y$ of codimension 1 with boundary $\partial P = B$ such that

1. X is transverse to the interior $\mathring{P}$,
2. X is tangent to the boundary B, i.e. B is X-invariant,
3. for each p in Y, there exists $t_+, t_- > 0$ such that

$$Fl^X_{t_+}(p),\ Fl^X_{-t_-}(p) \in P.$$

Global Hypersurface of Section

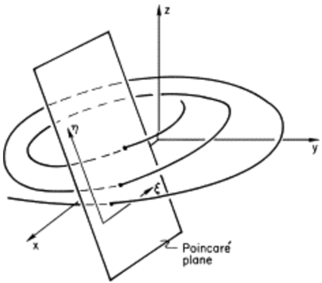


Figure 1: A picture of GHS in dimension 3 [1]

Intuition: hypersurface where every trajectory crosses

[1] https://www.sciencedirect.com/topics/engineering/poincare-section

Return Map

P: GHS of X on Y

We can define **(first) return time** τ_p for each $p \in \mathring{P}$ by

$$\tau_p = \min\{t > 0 : Fl_t^X(p) \in P\}$$

and the **return map** on $\mathring{P}$

$$\Psi(p) = Fl_{\tau_p}^X(p).$$

Note. Ψ is a diffeomorphism on $\mathring{P}$.

Generally, Ψ does not extend to the boundary.

Open Book Decomposition

Y: Closed manifold, X: Vector field on Y

A codimension 2 submanifold B with $\pi : Y \setminus B \to S^1$ defines **open book decomposition** (OBD) on Y adapted to X if

1. The normal bundle of B is trivial.
2. The map π is a fiber bundle such that $\pi(b; r, \theta) = e^{i\theta}$ on $\nu(B) \setminus B$.
3. X is transverse to each fiber $\pi^{-1}(\theta)$ and tangent to B.

$P_\theta = \overline{\pi^{-1}(\theta)}$: **page**, $B = \partial P_\theta$: **binding**

Open Book Decomposition

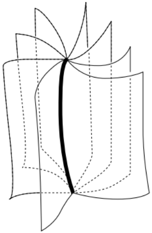


Figure 2: Illustration of OBD [2]

1. If (B, π) is an OBD of (Y, X), each page P_θ is a candidate for the GHS. We only need to check the boundedness of the return time.
2. If (B, π) is an OBD of a Reeb vector field of $(Y, \ker \alpha)$, then $(\mathring{P}, d\alpha)$ is a symplectic manifold and $(B, \xi_B = \xi|_{TB})$ is a contact manifold.
3. The return map is a symplectomorphism since $\Psi^* \alpha - \alpha = d\tau$.

[2] M. Kwon, O. van Koert "Brieskorn manifolds in contact topology"

Why GHS is useful?

Vector field X on Y^n: Dynamics of 1-parameter family of diffeomorphisms.

Return map Ψ on P^{n-1}: Dynamics of one diffeo/symplecto-morphism.

Ex. Periodic orbit of $X \Leftrightarrow$ Fixed point or periodic point of Ψ.

Finding periodic Reeb orbit is very important topic in symplectic geometry.

Conjecture (Weinstein Conjecture)

There exists at least one periodic Reeb orbit on compact contact manifold.

This is also related to finding periodic geodesics.

Birkhoff's Annulus

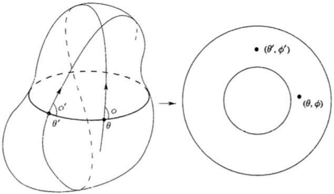


Figure 3: Birkhoff's annulus [3]

1. There exists at least one periodic geodesic γ on S^2.
   $\Rightarrow$ We take this orbit as an equator $q_1 = 0$.
2. If the curvature is positive, A is a GHS.

$$A = \{(q,p)\,:\,q_1 = 0,\,p_1 \geq 0\} \simeq S^1 \times [0,\pi]$$

p_1 corresponds to the angle between the orbit and γ.

[3] B. Cipra, D. Mackenzie "What's Happening in the Mathematical Sciences"

Birkhoff's Annulus

Theorem (Franks)

Any area-preserving homeomorphism of annulus has 0 of infinitely many fixed points.

$\Rightarrow S^2$ has infinitely many closed geodesics.

Hofer, Wysocki, Zehnder proved that there exists 2 or infinitely many periodic Reeb orbits on dynamically convex S^3, using GHS.

Difficulty in higher dimension: unstability of the boundary (codimension 2 invariant submanifold)

Ex. If X_H is a geodesic vector field, we need totally geodesic submanifold of codimension 1.

Mechanical Hamiltonian

Main Theorem 1

Theorem

Let $H : T^*\mathbb{R}^n \to \mathbb{R}$ be a symmetric mechanical Hamiltonian of a convex type. Then there exists a global hypersurface of section of Hamiltonian flow on $H^{-1}(c)$ given by

$$P = \left\{(q,p) \in H^{-1}(c) \, : \, q_1 = 0, \, p_1 \geq 0\right\}.$$

Moreover, the return map extends to the boundary.

Symmetric Mechanical Hamiltonian of Convex Type

Mechanical Hamiltonian: $H : T^*\mathbb{R}^n \to \mathbb{R}$ of the form

$$H(q,p) = W(p) + V(q).$$

such that $p \cdot \nabla W > 0 \Rightarrow X = p \cdot \partial_p$ is transversal to $H^{-1}(c)$

$\Rightarrow X$ is Liouville and $i_X\omega = pdq$ is a contact form on $H^{-1}(c)$.

Ex. $W(p) = |p|^2/2$: standard definition of mechanical Hamiltonian.

$\rho : \mathbb{R}^n \to \mathbb{R}^n$ be a reflection along a hyperplane containing 0.

H is **symmetric** if $V(q) = V(\rho(q))$ and $W(p) = W(\rho(p))$

We assume ρ is given by $\rho(v_1, \vec{v}) = \rho(-v_1, \vec{v})$.

Symmetric Mechanical Hamiltonian of Convex Type

Hill's region : c be a regular value of H. Hill's region is

$$\mathcal{H}_c^q = \mathrm{pr}_1(H^{-1}(c)) \subset \mathbb{R}^n \qquad \mathcal{H}_c^p = \mathrm{pr}_2(H^{-1}(c)) \subset \mathbb{R}^n$$

If $(q, p) \in H^{-1}(c)$, then $q \in \mathcal{H}_c^q$ and $p \in \mathcal{H}_c^p$.

Intuitively, $\mathcal{H}_c^q$ is the maximal region allowed for q-coordinate.

A symmetric mechanical Hamiltonian H is **of convex type** if

1. $\partial_{q_1}^2 V > 0$ and $\partial_{p_1}^2 W > 0$. (q_1, p_1 are the directions of reflection.)
2. $\mathcal{H}_c^p$ and $\mathcal{H}_c^q$ are compact.

Binding

Let $Y = H^{-1}(c)$ be a regular level set.

Hamiltonian vector field : $X_H = \nabla W \cdot \partial_q - \nabla V \cdot \partial_p$.

Define

$$B = \{(q,p) \in H^{-1}(c) \, : \, q_1 = p_1 = 0\}.$$

$V(q_1, \vec{q}) = V(-q_1, \vec{q})$, $W(p_1, \vec{p}) = W(-p_1, \vec{p})$ (Symmetry condition)

$\Rightarrow \partial_{q_1} V = \partial_{p_1} W = 0$ along B, so X_H is tangent to B.

$\Rightarrow B$ will be the binding of OBD, or the boundary of GHS.

Fibration and Angular Form

Define fibration $\pi : Y \setminus B \to S^1 \subset \mathbb{C}$ by

$$\pi(q,p) = \frac{q_1 + ip_1}{|q_1 + ip_1|}.$$

The **angular form** is defined by

$$\Theta = i \cdot d \log \pi = \frac{p_1 dq_1 - q_1 dp_1}{q_1^2 + p_1^2} = \frac{\theta}{q_1^2 + p_1^2}.$$

1. $\Theta(X_H)$ measures angular velocity of X_H.
2. If $\Theta(X_H) > 0$, then X_H is transversal to each page.
3. If $\Theta(X_H) > \varepsilon > 0$, then the return time is bounded by $2\pi/\varepsilon$.

   $\Rightarrow$ Closure of every fiber $P_\theta = \{(q,p) \,:\, \mathrm{Arg}(q_1 + ip_1) = \theta\}$ is GHS.

   In the theorem, we chose $\theta = i$ so $q_1 = 0$, $p_1 \geq 0$.

Proof of the Existence of GHS

We have

$$\theta(X_H) = p_1 \partial_{p_1} W + q_1 \partial_{q_1} V.$$

Near $p_1 = q_1 = 0$, take Taylor expansion so that

$$\theta(X_H) = p_1^2 \partial_{p_1}^2 W + q_1^2 \partial_{q_1}^2 V + O(|q_1^2 + p_1^2|)$$

$\partial_{q_1}^2 V > 0$, $\partial_{p_1}^2 W > 0$ and the compactness of Hill's region gives

$$\theta(X_H) > \varepsilon(q_1^2 + p_1^2)$$

which gives the lower bound of $\Theta(X_H)$ near B.

Proof of the Existence of GHS

Outside a neighborhood of B, we have

$$q_1 \partial_{q_1} V = q_1^2 \frac{\partial_{q_1} V}{q_1}.$$

$\partial_{q_1} V(0, \vec{q}) = 0$, $\partial_{q_1}^2 V > 0 \Rightarrow \partial_{q_1} V / q_1 > 0$ for any $q_1 \neq 0$.
($\partial_{q_1} V = V_1$ increases along q_1, so $V_1 < 0$ if $q_1 < 0$ and $V_1 > 0$ if $q_1 > 0$.)

Compactness of Hill's region gives the lower bound of $\partial_{q_1} V / q_1$.

Similarly, we can bound $\partial_{p_1} W / p_1$, which gives the lower bound of $\Theta(X_H)$.

Extension of Return Map

$(H^{-1}(c) = Y, \xi = \ker \alpha)$: Regular level set of contact type

(B, π): Open book decomposition of R on Y

γ: Contractible periodic Reeb orbit contained in B

ν: Symplectic normal bundle, i.e. $\xi_B \oplus \nu = \xi$ along γ.

There exists a Riemannian metric such that

$$\mathrm{Hess} H = \nabla(dH|_\xi) = \begin{pmatrix} S_\xi & 0 \\ 0 & S_\nu \end{pmatrix}$$

where $S_\xi \in \xi_B^* \otimes \xi_B^*$, $S_\nu \in \nu^* \otimes \nu^*$. Call S_ν **normal Hessian**.

Lemma

If S_ν is positive definite, the return map on a page can be extended smoothly to the boundary.

Extension of Return Map

Symplectic normal frame: $(\partial_{p_1}, \partial_{q_1})$

Normal Hessian is given by

$$S_N = \begin{pmatrix} \partial^2_{p_1} W & 0 \\ 0 & \partial^2_{q_1} V \end{pmatrix}$$

This is positive definite, so the return map extends to the boundary.

Example - Mechanical Hamiltonians

Mechanical Hamiltonian is Hamiltonian of the form

$$H(q,p) = \frac{1}{2}|p|^2 + V(q)$$

If the followings hold, we can apply the theorem

1. $V(q_1, \vec{q}) = V(-q_1, \vec{q})$.
2. $\partial_{q_1}^2 V > 0$.
3. $\mathcal{H}_c^q$ is compact.

Example: Harmonic oscillator

$$H(q,p) = \frac{1}{2}|p|^2 + \frac{k}{2}|q|^2$$

Example - Ellipsoids

For any Hamiltonians of the form

$$H(q, p) = \sum a_i p_i^2 + \sum b_i q_i^2$$

where $a_i, b_i > 0$, we can apply the theorem.

The level set is a contact ellipsoid.

Example - Hénon–Heiles system

Hénon-Heiles system is defined by a Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 + V(q) = \frac{1}{2}|p|^2 + \left(\frac{1}{2}|q|^2 + (q_1^2 + q_2^2)q_3 - \frac{q_3^3}{3}\right).$$

The Hill's region is compact and $q_2 > -1/2$ if $c < 1/6$.

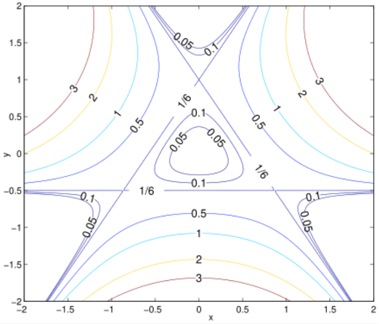


Figure 4: Contours of the Hénon-Heiles potential where $(x,y) = (\sqrt{q_1^2 + q_2^2}, q_3)$ [4]

[4]https://en.wikipedia.org/wiki/Hénon-Heiles_system

Example - Hénon–Heiles system

H describes the galactic dynamics, is non-integrable and chaotic.

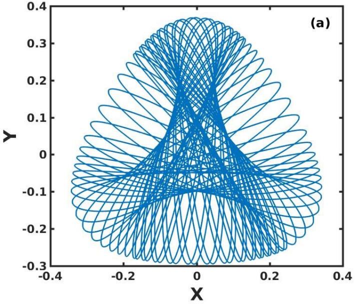


Figure 5: Orbit of the Hénon-Helies system [5]

$V(q_1, q_2, q_3) = V(-q_1, q_2, q_3)$ and $\partial^2_{q_1} V = 1 + 2q_3 > 0$, so GHS is

$$P = \left\{(q, p) \in H^{-1}(c) \,:\, q_1 = 0,\, p_1 \geq 0\right\}.$$

[5] A. Zafar, M. Khan "Energy behavior of Boris algorithm"

Convex Hypersurfaces

Geodesic Vector Field

Let (M, g) be a complete Riemannian manifold.

$\forall (q, p) \in TM$, $\exists!$ geodesic γ s.t. $\gamma(0) = q$, $\dot{\gamma}(0) = p$.

The exponential map

$$\begin{aligned} \exp : TM &\to TM \\ (q, p) &\mapsto (\gamma(1), \dot{\gamma}(1)) \end{aligned}$$

defines 1-parameter family of diffeomorphisms $\varphi_t(q, p) = \exp_q(tp)$, which is called **geodesic flow**.

Geodesic vector field is defined by

$$X_g(q, p) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t(q, p).$$

We define (co-)geodesic flow and (co-)geodesic vector field on T^*M by isomorphism between TM.

Geodesic Flow as a Hamiltonian Flow

Proposition

Let M be a Riemannian manifold. The geodesic vector field on T^*M is a Hamiltonian vector field with Hamiltonian

$$H(q,p) := \frac{1}{2}||p||_{g^*}^2$$

where T^*M is equipped with a canonical symplectic form $\omega = dp \wedge dq$. Moreover, X_H on $H^{-1}(1/2)$ is identical to the Reeb vector field of the unit cotangent bundle $(ST^*M, \ker(pdq))$.

Main Theorem 2

Theorem (Cho, L- '24)

Let $M \subset \mathbb{R}^{n+1}$ be a locally symmetric convex hypersurface with codimension 1 fixed locus N. Then the geodesic flow on ST^*M admits a global hypersurface of section

$$P = \{(x, y) \in ST^*M \,:\, x \in N,\, \langle y, \nu \rangle \geq 0\}$$

where ν is the normal vector of N. Moreover, the return map $\Psi : \mathring{P} \to \mathring{P}$ extends smoothly to the boundary of P.

This is a generalization of the Birkhoff's annulus.

(Closed geodesic $\Rightarrow$ Totally geodesic submanifold N)

Locally Symmetric Convex Hypersurface

$M = f^{-1}(c)$: regular level set in $\mathbb{R}^{n+1}$.

$N \subset M$: codimension 1 submanifold of M.

M is **locally symmetric with fixed locus** N
if there exists a reflection $\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ s.t.

1. $\rho|_N = \mathrm{Id}_N$, i.e. N is contained in the hypersurface fixed by ρ.
2. There exists a tubular neighborhood U_N of N s.t. $\rho(U_N) = U_N$.
   i.e. $\rho|_{U_N}$ is well-defined.

M is **convex** if M has positive sectional curvature.

Condition 1 : Symmetry

Lemma

Let (M, g) be a Riemannian manifold and N be a closed submanifold. Assume that there exist a tubular neighborhood $\nu(N)$ of N and a locally defined isometric involution $i : \nu(N) \to \nu(N)$ such that $i(N) = N$. Then N is a totally geodesic submanifold.

First condition (isometric reflection)

$\Rightarrow N$ is totally geodesic submanifold.

$\Rightarrow ST^*N$ is tangent to the geodesic vector field

$\Rightarrow ST^*N$ is the binding of OBD

Condition 1 : Symmetry

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $M = f^{-1}(c)$ is a regular level set.

Note. If $M \subset \mathbb{R}^{n+1}$ is an oriented submanifold of codimension 1, M is always a level set of a function.

For convenience, we assume that

$$f(x_0, \vec{x}) = f(-x_0, \vec{x}) \text{ for } |x_0| < \varepsilon,$$

i.e. the reflection is given by $\rho(x_0, \vec{x}) = (-x_0, \vec{x})$.

$\Rightarrow \{x_0 = 0\} \cap M = N$ is totally geodesic submanifold of M.

Condition 2 : Convexity

Lemma

The sectional curvature of regular level set $f^{-1}(c)$ is positive if and only if $\mathrm{Hess}(f)$ is positive definite or negative definite.

Second condition (positive sectional curvature)

$\Leftrightarrow \mathrm{Hess}(f)$ is positive/negative definite.
(If negative definite, we use $-f$ instead.)

Note. M bounds a convex domain, so $M \simeq S^n$ and $N \simeq S^{n-1}$.

Normal bundle of codimension 1 closed submanifold of S^n is trivial.

$\Rightarrow ST^*N$ has trivial normal bundle in ST^*M.

Explicit Formula for the Hamiltonian Vector Field

Lemma

Let $\tilde{H}$ be a Hamiltonian on W, and V be a symplectic submanifold given by $f^{-1}(c_1) \cap g^{-1}(c_2)$. Let $H = \tilde{H}|_V$, then

$$X_H = X_{\tilde{H}} - \frac{\{g, \tilde{H}\}}{\{g, f\}} X_f - \frac{\{f, \tilde{H}\}}{\{f, g\}} X_g.$$

where $\{f, g\}$ is a Poisson bracket.

$T^*M = \{(x, y) \in T^*\mathbb{R}^{n+1} \,:\, f(x) = c, \, \langle \nabla f(x), y \rangle = 0\}.$

$\Rightarrow f(x, y) = f(x), \, g(x, y) = \langle f(x), y \rangle, \, \tilde{H}(x, y) = |y|^2/2$

$$X_H = y \cdot \partial_x - \frac{\mathrm{Hess}(f)_x(y, y)}{\|\nabla f(x)\|^2} \nabla f \cdot \partial_y.$$

Angular Form

Let $M = f^{-1}(c)$, $Y = ST^*M$ and

$$B = \{(x, y) \in ST^*M \ : \ x_0 = y_0 = 0\} = ST^*N.$$

Define fibration $\pi : Y \setminus B \to S^1 \subset \mathbb{C}$ by

$$\pi(x, y) = \frac{x_0 + iy_0}{|x_0 + iy_0|}.$$

The angular form is defined by

$$\Theta = i \cdot d \log \pi = \frac{y_0 dx_0 - x_0 dy_0}{x_0^2 + y_0^2} = \frac{\theta}{x_0^2 + y_0^2}.$$

Bounding the Angular Form

Put X_H into θ, we have

$$\theta(X_H) = y_0^2 + x_0^2 \frac{\mathrm{Hess}(f)_x(y,y)}{\|\nabla f(x)\|^2} \frac{f_0(x)}{x_0}.$$

Positive-definiteness of $\mathrm{Hess}(f)$ and compactness of Y

$\Rightarrow$ lower bound of $\mathrm{Hess}(f)(y,y)$ and $f_{00}(x) = \mathrm{Hess}(f)_x((y_0,0),(y_0,0))$.

This gives the positive lower bound of $\theta(X_H)$.

Extension of Return Map

Symplectic normal frame: $(\partial_{y_0}, \partial_{x_0})$

Normal Hessian is given by

$$S_N = \operatorname{diag}\left(1, \frac{\operatorname{Hess}(f)_x(y,y)}{||\nabla f(x)||^2} f_{00}(x)\right).$$

Positive definiteness of $\operatorname{Hess}(f)$ gives the positive definiteness of S_N.

Topology of GHS

1. $M = f^{-1}(c) \simeq S^n$: M bounds a compact convex domain in $\mathbb{R}^{n+1}$
2. $Y = ST^*M \simeq ST^*S^n$
3. $N \simeq S^{n-1}$ is an *equator* of M
4. $P \simeq DT^*S^{n-1}$ is a unit upper-hemisphere bundle of S^{n-1}
5. $B \simeq ST^*S^{n-1}$

Example - Hypersurface of Revolution

Assume $||\vec{x}|| = ||\vec{x}\,'|| \Rightarrow f(x_0, \vec{x}) = f(x_0, \vec{x}\,')$.

We can apply the theorem if f is symmetric and convex, and get GHS

$$P = \{(0, \vec{x}; y_0, \vec{y}) : y_0 \geq 0\}.$$

Parametrization of $M = f^{-1}(c)$ is

$$M \cap \{(x_0, x_1, 0, \cdots, 0)\} = \{(a(\phi), \alpha \cos \phi, 0, \cdots, 0) : \phi \in \mathbb{R}\}$$

$\Rightarrow$ Explicit computation of the return map.

Example - Hypersurface of Revolution

Proposition (Cho, L- '24)

The return map $\Psi : P \to P$ *is given by*

$$\Psi\left((0,\vec{x}),(y_0,\vec{y})\right) = \Bigg(\left(0, \cos G(\|\vec{y}\|)\vec{x} + \frac{\alpha}{\|\vec{y}\|}\sin G(\|\vec{y}\|)\vec{y}\right), \left(y_0, -\frac{\|\vec{y}\|}{\alpha}\sin G(\|\vec{y}\|)\vec{x} + \cos G(\|\vec{y}\|)\vec{y}\right)\Bigg)$$

where

$$G(t) := t\int_0^{2\pi} \frac{\sqrt{\alpha^2(1-t^2)\sin^2\sigma + \{a'(\arcsin(\sqrt{1-t^2}\sin\sigma))\}^2}}{\alpha(1-(1-t^2)\sin^2\sigma)} d\sigma.$$

if $t \neq 0$, *and* $G(0) = 2\pi$.

Ellipsoid case : G consists of elliptic integrals

Example - Kepler Problem

Kepler Hamiltonian on $T^*\mathbb{R}^3 \setminus \{0\}$: $E(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$

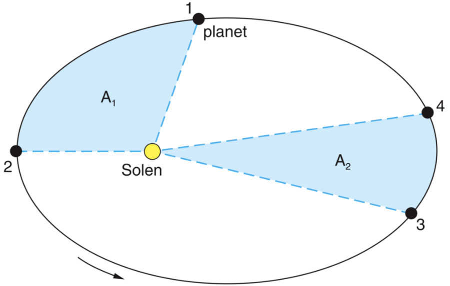


Figure 6: Kepler orbit [6]

The Hamiltonian has a singularity and Hill's region $\mathcal{H}_c^p$ is noncompact.

These problems can be solved by **Moser regularization**.

[6] https://snl.no/Keplers_problem

Moser Regularization

Fix $E = E_0$. Consider the Hamiltonian on $T^*\mathbb{R}^3$

$$\tilde{K}(q,p) = \frac{1}{2}\left(|q|\left(E(q,p) - E_0\right) + 1\right)^2 = \frac{1}{2}\left(\frac{1}{2}(|p|^2 - 2E_0)|q|\right)^2.$$

This is the Hamiltonian of geodesic vector field on $T^*S^3_r$ under the stereographic projection

$$\Phi_r : T^*S^3_r \to T^*\mathbb{R}^3$$
$$(x,y) \mapsto \left(\frac{r\vec{x}}{r - x_0}, \frac{r - x_0}{r}\vec{y} + \frac{y_0}{r}\vec{x}\right)$$

where $r = \sqrt{-2E_0}$, composed with a switch map $(q,p) \mapsto (-p,q)$.

Moser Regularization

On $T_r^*S^3$, we have

$$K_r = \frac{r^4}{2}|y|^2$$

The level set $E^{-1}(E_0)$ can be embedded into $K^{-1}(1/2)$

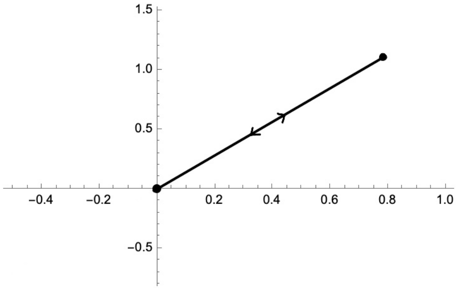
$\Rightarrow$ Kepler Hamiltonian vector field and geodesic vector field are parallel.

$$X_K|_{E^{-1}(E_0)}(p,q) = |q| X_E|_{E^{-1}(E_0)}(-q,p).$$

We regard Kepler problem as a sub-system of the geodesic flow on T^*S^3.

Collision Orbits

The singularity is normalized by adding **collision orbits**, which is the orbit bounces back from the origin.



Collision Orbits

In $T^*\mathbb{R}^3$, the (reparametrized) collision orbit is given by

$$(q(t), p(t)) = \left(0, 0, \frac{1}{r^2}(1 + \cos(rt)), 0, 0, -\frac{r\sin(rt)}{1 + \cos(rt)}\right).$$

This is the image of a great circle

$$\gamma(t) = (-\cos(rt), 0, 0, -\sin(rt); \sin(rt)/r, 0, 0, -\cos(rt)/r)$$

under the stereographic projection.

$t = 0 : (q_3, p_3) = (2/r^2, 0)$ the highest point

$t \to \pi/r- : (q_3, p_3) = (0, -\infty)$ collides into the origin

$t \to \pi/r+ : (q_3, p_3) = (0, \infty)$ bounce back

$t = 2\pi : (q_3, p_3) = (2/r^2, 0)$ the highest point

GHS of Kepler Problem

On $ST^*S^3_r$, we can directly apply the theorem and get GHS

$$P = \left\{(x, y) \in ST^*S^3 \,:\, x_3 = 0,\, y_3 \geq 0\right\}.$$

On $T^*\mathbb{R}^3$, we have

$$x_3 = \frac{2r^2 p_3}{p^2 + r^2} = 0, \qquad y_3 = \frac{p^2 + r^2}{2r^2} q_3 - \frac{p \cdot q}{r^2} p_3 \geq 0.$$

The GHS is given by $P = \left\{(q, p) \in T^*\mathbb{R}^3 \,:\, q_3 \geq 0,\, p_3 = 0\right\}$.

P is the set of highest points of the orbits.

Every orbit reaches maximum ($p_3 = 0$) on the upper half-plane ($q_3 \geq 0$).

In particular, the binding ∂P is planar Kepler problem.

Note. Other choice of P_θ gives complicated formula.

Closing

Further Discussions

1. (Case 1) Finding more examples with physical significance.
2. (Case 2) Extend the result to Finsler geodesic flows.
   cf. rotating Kepler problem, restricted 3-body problem
3. (Case 2) Extend the result to geodesic flows on δ-pinched spaces.
   cf. sphere theorem
4. (Case 2) Systolic inequality using the computed return map.